New results and conjectures on 2-partitions of multisets.

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***Abstract*—The interplay between integer sequences and partitions has led to numerous interesting results, with implications in generating functions, integral formulae, or combinatorics. An illustrative example is the number of solutions at level n to the *signum equation*. Denoted by $S(n)$, this represents the number of ways of choosing $+$ and $-$ such that $\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0$ (see A063865 in OEIS). The Andrica-Tomescu conjecture regarding the asymptotic behaviour of $S(n)$ was solved affirmatively in 2013, and new conjectures were formulated since then. In this paper we present recurrence formulae, generating functions and integral formulae for the number of ordered 2-partitions of the multiset M having equal sums. Certain related integer sequences not currently indexed in the OEIS are then presented. Finally, we formulate conjectures regarding the unimodality, distribution and asymptotic behaviour of these sequences.**

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I. Introduction

The *signum equation* is an old combinatorial problem considered by S.Finch [7]. For a given positive integer n, the level n solution of this equation represents the number of ways of choosing $+$ and $-$ such that $\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0$. Denoted by $S(n)$, this also represents the number of partitions of $\{1, 2, \ldots, n\}$ in two sets with equal sums. Sequence $\{S(n)\}_{n \geq 0}$ is indexed as A063865 in the Online Encyclopedia of Integer Sequences (OEIS) [9], and its first few terms are

$$1, 0, 0, 2, 2, 0, 0, 8, 14, 0, 0, 70, 124, 0, 0, 722, 1314, 0, 0, \ldots$$

For example, S(7)=8 since $1 + \cdots + 7 = 28$ and

$$14 = 1 + 6 + 7 = 2 + 5 + 7 = 3 + 4 + 7 = 3 + 5 + 6.$$

In 2002, Andrica and Tomescu [3] conjectured the following asymptotic formula for $S(n)$:

$$\lim_{\substack{n \to \infty \\ n \equiv 0 \ or \ 3 (mod 4)}} \frac{S(n)}{\frac{2^n}{n\sqrt{n}}} = \sqrt{\frac{6}{\pi}}.$$

Referred to as the Andrica-Tomescu conjecture, a possible analytical proof was suggested in [8]. A complete proof was for the first time obtained in 2013 by Sullivan [11], who used powerful analytic methods. Details of the proof and some comments on possible extensions using a strong version of the Central Limit Theorem can be found in [2].

Starting from an interesting analysis problem [1], Andrica established a generating function which allowed novel approaches in the study of 2-partitions with equal sums for multisets. For more information regarding the general theory of multisets one can consult the article of Stanley [10]. The connection with unimodal polynomials is presented in [4] and with some aspects in special representations of integers, known as Erdös-Suranyi representations are given in [5],[6] and [7].

The paper is structured as follows. In Section II we introduce $S(m_1, ..., m_k; \alpha_1, ..., \alpha_k)$, the number of ordered 2-partitions of the multiset M having equal sums and present the generating function and a useful integral formula. The multisets with equal multiplicities and the connection with the Laurent ring $\mathbb{Z}[X, X^{-1}]$ are studied in Section III where some recurrence formulae are also given. New integer sequences are derived in Section IV, for small values of the multiplicities. Section V presents some conjectures concerning the asymptotic behaviour, unimodality and statistical distribution.

II. 2-partitions of multisets

Let $\alpha_1, ..., \alpha_k$ be real numbers, let m_1, ..., m_k be positive integers, and let M be the multiset

$$M = \{\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1 \ times}, \cdots, \underbrace{\alpha_k, \cdots, \alpha_k}_{m_k \ times}\}.$$

We call m_1, ..., m_k the *multiplicities* of the elements in the multiset M. Denote by $S(m_1, ..., m_k; \alpha_1, ..., \alpha_k)$ the number of ordered 2-partitions of M having equal sums, i.e., the number of pairs (C_1, C_2) of subsets of M such that

(i) $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \emptyset$

(ii) $\sum_{x \in C_1} x = \sum_{x \in C_2} x = \frac{1}{2} \sum_{i=1}^{k} m_i \alpha_i$.

If we assume that M is a multiset of positive integers, then $S(m_1, ..., m_k; \alpha_1, ..., \alpha_k)$ is the constant term in the expansion

$$F(z) = \left(z^{\alpha_1} + \frac{1}{z^{\alpha_1}}\right)^{m_1} \left(z^{\alpha_2} + \frac{1}{z^{\alpha_2}}\right)^{m_2} \cdots \left(z^{\alpha_k} + \frac{1}{z^{\alpha_k}}\right)^{m_k}. \quad (1)$$

Let us observe that if we expand $F(z)$, we get

$$F(z) = c_0(m_1, ..., m_k; \alpha_1, ..., \alpha_k) + \sum_{j\in\mathbb{Z}\setminus\{0\}} c_j z^j, \quad (2)$$

where, for the unity of notation, we use $c_0(m_1, ..., m_k; \alpha_1, ..., \alpha_k) = S(m_1, ..., m_k; \alpha_1, ..., \alpha_k)$ and $c_j = c_j(m_1, ..., m_k; \alpha_1, ..., \alpha_k) \in \mathbb{Z}$. If we set $z = \cos t + i \sin t$, in (1) and (2), we get the equivalent form

$$2^{m_1+\cdots+m_k} \prod_{s=1}^{k} (\cos \alpha_s t)^{m_s} = c_0(m_1, ..., m_k; \alpha_1, ..., \alpha_k) + \sum_{j\in\mathbb{Z}\setminus\{0\}} c_j (\cos jt + i \sin jt).$$

Integrating this last identity over the interval $[0, 2\pi]$, we obtain the following integral formula for the number $c_0(m_1, ..., m_k; \alpha_1, ..., \alpha_k)$:

$$c_0(m_1, ..., m_k; \alpha_1, ..., \alpha_k) = \frac{2^{m_1+\cdots+m_k}}{2\pi} \int_0^{2\pi} \prod_{s=1}^{k} (\cos \alpha_s t)^{m_s} dt. \quad (3)$$

From formulae (1) and (2) it is clear that the coefficient $c_j(m_1, ..., m_k; \alpha_1, ..., \alpha_k)$ has a combinatorial interpretation: it is the number of representations of integer j as

$$j = \underbrace{\pm\alpha_1 \pm \cdots \pm \alpha_1}_{m_1\ times} \pm \cdots \pm \underbrace{\alpha_k \pm \cdots \pm \alpha_k}_{m_k\ times},$$

for all possible $2^{m_1+\cdots+m_k}$ choices of signs $+$ and $-$.

If we multiply the relation (2) by z^{-j} we get

$$z^{-j} F(z) = c_j(m_1, ..., m_k; \alpha_1, ..., \alpha_k) + \sum_{l\neq j} c_l z^l, \quad (4)$$

Considering $z = \cos t + i \sin t$ in (4), and integrating over the interval $[0, 2\pi]$, we obtain the following integral formula for the number $c_j(m_1, ..., m_k; \alpha_1, ..., \alpha_k)$:

$$c_j(m_1, ..., m_k; \alpha_1, ..., \alpha_k) = \frac{2^{m_1+\cdots+m_k}}{2\pi} \int_0^{2\pi} \cos jt \prod_{s=1}^{k} (\cos \alpha_s t)^{m_s} dt. \quad (5)$$

□

III. Multisets with the same multiplicities and the Laurent ring $\mathbb{Z}[X, X^{-1}]$

Recall that a Laurent polynomial with integer coefficients is an expression of the form

$$p = \sum_{k\in\mathbb{Z}} a_k X^k, \quad a_k \in \mathbb{Z}$$

where X is a formal variable, and only finitely many coefficients a_k are non-zero. Two Laurent polynomials are equal if their coefficients are equal. Such expressions can be added, multiplied, and brought back to the same form by reducing the corresponding similar terms.

Formulae for addition and multiplication are the same as for the ordinary polynomials, with the only difference being that both positive and negative powers of X can be present. The set of Laurent polynomials $\mathbb{Z}[X, X^{-1}]$ is a ring with respect to the addition and multiplication.

A Laurent polynomial $p \in \mathbb{Z}[X, X^{-1}]$ is *symmetric* if it satisfies the relation $p(X) = p(X^{-1})$. This property is equivalent to $a_{-k} = a_k$, for all $k \in \mathbb{Z}$. The maximal positive integer s with $a_s \neq 0$ defines the degree of p. The set of all symmetric Laurent polynomials $\mathbb{Z}_{\text{symm}}[X, X^{-1}]$ is a subring of $\mathbb{Z}[X, X^{-1}]$.

Clearly, from the relation $F(z) = F(1/z)$ it follows that the expression of $F(z)$ in (1), after we expand, is a symmetric Laurent polynomial of variable z and degree $\sum_{i=1}^{k} m_i \alpha_i$.

When $m_1 = ... = m_k = m$, we say that the elements of the multiset M are of the multiplicity m.

In this paper we are interested in the symmetric Laurent polynomials generated by the expansion

$$F_{n,m}(z) = \left(z + \frac{1}{z}\right)^m \left(z^2 + \frac{1}{z^2}\right)^m \cdots \left(z^n + \frac{1}{z^n}\right)^m,$$

that is the situation $k = n$ and $\alpha_1 = 1, \ldots, \alpha_n = n$. In this case the Laurent polynomial $F_{n,m}(z)$ has the degree $\frac{mn(n+1)}{2}$ and it is of the form

$$F_{n,m}(z) = \sum_{j\in\mathbb{Z}} c_j^{(m)}(n) z^j.$$

A. Recurrence formulae

If we multiply the polynomials $F_{n,k}(z)$ and $F_{n,l}(z)$, we have the following formula for the coefficients of $F_{n,k+l}(z)$

$$c_d^{(k+l)}(n) = \sum_{j\in\mathbb{Z}} c_{j+d}^{(k)}(n) c_j^{(l)}(n), \quad d \in \mathbb{Z}. \quad (6)$$

Recurrent formulae for $c_j^{(m)}(n)$ as a function of terms of the form $c_j^{(m)}(n)$, which allow an efficient numerical computation.

For $m = 1$ one can obtain $c_j^{(1)}(n)$ recursively

$$\begin{aligned} F_{n,1}(z) &= \sum_{j\in\mathbb{Z}} c_j^{(1)}(n) z^j \\ &= F_{n-1,1}(z) \left(z^n + \frac{1}{z^n}\right) \\ &= \left(\sum_{j\in\mathbb{Z}} c_j^{(1)}(n-1) z^j\right) \left(z^n + \frac{1}{z^n}\right) \\ &= \left(\sum_{j\in\mathbb{Z}} c_j^{(1)}(n-1) z^{j+n}\right) + \left(\sum_{j\in\mathbb{Z}} c_j^{(1)}(n-1) z^{j-n}\right) \\ &= \sum_{j\in\mathbb{Z}} \left(c_{j-n}^{(1)}(n-1) + c_{j+n}^{(1)}(n-1)\right) z^j. \end{aligned}$$

Hence,

$$c_j^{(1)}(n) = c_{j-n}^{(1)}(n-1) + c_{j+n}^{(1)}(n-1), \text{ for } j \in \mathbb{Z}. \quad (7)$$

Similarly, $c_j^{(2)}(n)$ can also be obtained recurrently by

$$\begin{aligned}
F_{n,2}(z) &= \sum_{j\in\mathbb{Z}} c_j^{(2)}(n) z^j \\
&= F_{n-1,2}(z)\left(z^n + \frac{1}{z^n}\right)^2 \\
&= \left(\sum_{j\in\mathbb{Z}} c_j^{(2)}(n-1) z^j\right)\left(z^{2n} + 2 + \frac{1}{z^{2n}}\right) \\
&= \sum_{j\in\mathbb{Z}} c_j^{(2)}(n-1) z^{j+2n} \\
&+ \sum_{j\in\mathbb{Z}} c_j^{(2)}(n-1) z^j + \sum_{j\in\mathbb{Z}} c_j^{(2)}(n-1) z^{j-2n} \\
&= \sum_{j\in\mathbb{Z}} \left(c_{j-2n}^{(2)}(n-1) + 2c_j^{(2)}(n-1) + c_{j+2n}^{(2)}(n-1)\right) z^j.
\end{aligned}$$

Hence, for $j \in \mathbb{Z}$ one obtains

$$c_j^{(2)}(n) = c_{j-2n}^{(2)}(n-1) + 2c_j^{(2)}(n-1) + c_{j+2n}^{(2)}(n-1). \quad (8)$$

The following recurrences are valid for $j \in \mathbb{Z}$ and $n \geq 1$.

$$\begin{aligned}
c_j^{(3)}(n) &= c_{j-3n}^{(3)}(n-1) + 3c_{j-n}^{(3)}(n-1) \\
&\quad + 3c_{j+n}^{(3)}(n-1) + c_{j+3n}^{(3)}(n-1) \qquad (9) \\
c_j^{(4)}(n) &= c_{j-4n}^{(4)}(n-1) + 4c_{j-2n}^{(4)}(n-1) + 6c_j^{(4)}(n-1) \\
&\quad + 4c_{j+2n}^{(4)}(n-1) + c_{j+4n}^{(4)}(n-1). \qquad (10)
\end{aligned}$$

In general, for m even one obtains the formula

$$c_j^{(m)}(n) = \sum_{k=1}^{m/2} \binom{m}{k} c_{j\pm 2kn}^{(m)}(n-1) + \binom{m}{m/2} c_j^{(m)}(n-1), \quad (11)$$

$$c_j^{(m)}(n) = \sum_{k=1}^{(m+1)/2} \binom{m}{2k-1} c_{j\pm(2k-1)n}^{(m)}(n-1). \quad (12)$$

when m is even or odd, respectively.

IV. Associated integer sequences

Various integer sequences can be recovered from the coefficients of $F_{n,m}(z)$. Of particular interest are the sequences $c_0^{(m)}(n)$, representing the constant term of $F_{n,m}(z)$. As seen in Section II, the sequences $c_j^{(m)}(n)$, $j = 1, 2, \ldots$ have an interesting combinatorial interpretation and importance.

A. *The case $m = 1$*

For $m = 1$, sequence $c_0^{(1)}(n)$ represents also the number of solutions at level n to the *signum equation*, which is A063865 in OEIS. More precisely, $c_0^{(1)}(n)$ is the number of ways of choosing $+$ and $-$ such that $\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0$, sometimes denoted by $S(n) = c_0^{(1)}(n)$. This is in fact an old combinatorial problem also considered by S.Finch [7]. As an example, we have $c_0^{(1)}(40) = 5830034720$ and $c_0^{(1)}(100) = 173102400594872501663378632 4$.

Concerning its asymptotic behaviour, Andrica and Tomescu [3] conjectured the following asymptotic formula in 2002:

$$\lim_{\substack{n\to\infty \\ n\equiv 0 \text{ or } 3 (mod 4)}} \frac{S(n)}{\frac{2^n}{n\sqrt{n}}} = \sqrt{\frac{6}{\pi}}.$$

This was proved analytically in 2013 by B.D. Sullivan [11].

For $m = 1$ and fixed values of n, the coefficients $\{c_j^{(1)}(n)\}_{j\geq 0}$ produce the following finite sequences:

$c_j^{(1)}(1)$:	$0, 1$
$c_j^{(1)}(2)$:	$0, 1, 0, 1$
$c_j^{(1)}(3)$:	$2, 0, 1, 0, 1, 0, 1$
$c_j^{(1)}(4)$:	$2, 0, 2, 0, 2, 0, 1, 0, 1, 0, 1$
$c_j^{(1)}(5)$:	$0, 3, 0, 3, 0, 3, 0, 2, 0, 2, 0, 1, 0, 1, 0, 1$
$c_j^{(1)}(6)$:	$0, 5, 0, 5, 0, 4, 0, 4, 0, 4, 0, 3, 0, 2, 0, 2, 0, 1, 0, 1, 0, 1$
$c_j^{(1)}(7)$:	$8, 0, 8, 0, 8, 0, 7, 0, 7, 0, 6, 0, 5, 0, 5, 0, 4, 0, 3, 0, 2, 0, 2, 0, 1, 0, 1, 0, 1$
$c_j^{(1)}(8)$:	$14, 0, 13, 0, 13, 0, 13, 0, 12, 0, 11, 0, 10, 0, 9, 0, 8, 0, 7, 0, 6, 0, 5, 0, 4, 0, 3, 0, 2, 0, 2, 0, 1, 0, 1, 0, 1$
$c_j^{(1)}(9)$:	$0, 23, 0, 23, 0, 22, 0, 21, 0, 21, 0, 19, 0, 18, 0, 17, 0, 15, 0, 13, 0, 12, 0, 10, 0, 9, 0, 8, 0, 6, 0, 5, 0, 4, 0, 3, 0, 2, 0, 2, 0, 1, 0, 1, 0, 1$
$c_j^{(1)}(10)$:	$0, 40, 0, 39, 0, 39, 0, 38, 0, 36, 0, 35, 0, 33, 0, 31, 0, 29, 0, 27, 0, 24, 0, 22, 0, 20, 0, 17, 0, 15, 0, 13, 0, 11, 0, 10, 0, 8, 0, 6, 0, 5, 0, 4, 0, 3, 0, 2, 0, 2, 0, 1, 0, 1, 0, 1.$

One can notice that $\{c_j^{(1)}(n)\}_{j\geq 0}$ has $n(n+1)/2+1$ relevant terms, i.e., $c_j^{(1)}(n) = 0$ for $j > n(n+1)/2+1$. The rows of this triangle are not indexed in the OEIS.

B. *The case $m = 2$*

For $m = 2$ one recovers the sequence

$$c_0^{(2)}(n) : 2, 4, 10, 26, 76, 236, 760, 2522, 8556, 29504, \ldots$$

indexed in OEIS as A047653. By the recurrence formula (6), it can be be shown that

$$c_0^{(2)}(n) = \sum_{j\in\mathbb{Z}} c_j^{(1)}(n) c_j^{(1)}(n) = S^2(n) + 2\sum_{j=1}^{n(n+1)/2} (c_j^{(1)}(n))^2. \quad (13)$$

Concerning its asymptotic equivalent, the following formula was conjectured by Kotesovec in 2014 (see A047653 in [9])

$$\lim_{n\to\infty} \frac{c_0^{(2)}(n)}{\frac{4^n}{n\sqrt{n}}} = \sqrt{\frac{3}{\pi}}.$$

For $m = 2$ and fixed values of n, the coefficients $\{c_j^{(2)}(n)\}_{j\geq 0}$ generate the following finite sequences:

$c_j^{(2)}(1):$ $2, 0, 1$

$c_j^{(2)}(2):$ $4, 0, 3, 0, 2, 0, 1$

$c_j^{(2)}(3):$ $10, 0, 8, 0, 7, 0, 6, 0, 3, 0, 2, 0, 1$

$c_j^{(2)}(4):$ $26, 0, 24, 0, 22, 0, 20, 0, 16, 0, 12, 0, 9, 0, 6, 0, 3, 0, 2, 0, 1$

$c_j^{(2)}(5):$ $76, 0, 73, 0, 70, 0, 65, 0, 58, 0, 51, 0, 42, 0, 34, 0, 26, 0, 20, 0, 14, 0, 9, 0, 6, 0, 3, 0, 2, 0, 1$

$c_j^{(2)}(6):$ $236, 0, 231, 0, 224, 0, 215, 0, 200, 0, 184, 0, 166, 0, 144, 0, 124, 0, 106, 0, 86, 0, 69, 0, 54, 0, 40, 0, 30, 0, 22, 0, 14, 0, 9, 0, 6, 0, 3, 0, 2, 0, 1$

$c_j^{(2)}(7):$ $760, 0, 752, 0, 738, 0, 716, 0, 684, 0, 646, 0, 603, 0, 554, 0, 501, 0, 450, 0, 396, 0, 344, 0, 295, 0, 248, 0, 205, 0, 168, 0, 134, 0, 104, 0, 81, 0, 60, 0, 44, 0, 32, 0, 22, 0, 14, 0, 9, 0, 6, 0, 3, 0, 2, 0, 1$

$c_j^{(2)}(8):$ $2522, 0, 2508, 0, 2475, 0, 2422, 0, 2347, 0, 2256, 0, 2149, 0, 2028, 0, 1896, 0, 1756, 0, 1611, 0, 1464, 0, 1318, 0, 1174, 0, 1035, 0, 904, 0, 778, 0, 664, 0, 561, 0, 466, 0, 384, 0, 312, 0, 249, 0, 196, 0, 152, 0, 116, 0, 87, 0, 64, 0, 46, 0, 32, 0, 22, 0, 14, 0, 9, 0, 6, 0, 3, 0, 2, 0, 1$

$c_j^{(2)}(9):$ $8556, 0, 8523, 0, 8442, 0, 8311, 0, 8124, 0, 7894, 0, 7624, 0, 7309, 0, 6964, 0, 6595, 0, 6196, 0, 5787, 0, 5370, 0, 4944, 0, 4522, 0, 4109, 0, 3700, 0, 3311, 0, 2942, 0, 2589, 0, 2264, 0, 1964, 0, 1686, 0, 1436, 0, 1214, 0, 1013, 0, 840, 0, 690, 0, 558, 0, 448, 0, 356, 0, 277, 0, 214, 0, 164, 0, 122, 0, 91, 0, 66, 0, 46, 0, 32, 0, 22, 0, 14, 0, 9, 0, 6, 0, 3, 0, 2, 0, 1.$

Notice that $\{c_j^{(2)}(n)\}_{j\geq 0}$ has $n(n+1)+1$ relevant terms, i.e., $c_j^{(2)}(n) = 0$ for $j > n(n+1)+1$. The rows of this triangle are not indexed in the OEIS.

C. *The case* $m = 3$

For $m = 3$ one obtains the sequence

$$c_0^{(3)}(n) : 0, 0, 62, 332, 0, 0, 80006, 531524, 0, 0, 173607568, \ldots$$

indexed in OEIS as A124995. While several terms of this sequences have been computed, its asymptotic behaviour is not currently known. A conjecture based on numerical calculations is formulated in the following section.

For $m = 3$ and fixed values of n, the coefficients $\{c_j^{(3)}(n)\}_{j\geq 0}$ produce the following finite sequences:

$c_j^{(3)}(1):$ $0, 3, 0, 1$

$c_j^{(3)}(2):$ $0, 12, 0, 10, 0, 6, 0, 3, 0, 1$

$c_j^{(3)}(3):$ $62, 0, 57, 0, 51, 0, 43, 0, 30, 0, 21, 0, 13, 0, 6, 0, 3, 0, 1$

$c_j^{(3)}(4):$ $332, 0, 327, 0, 309, 0, 278, 0, 243, 0, 204, 0, 161, 0, 123, 0, 90, 0, 61, 0, 39, 0, 24, 0, 13, 0, 6, 0, 3, 0, 1$

$c_j^{(3)}(5):$ $0, 1974, 0, 1932, 0, 1851, 0, 1731, 0, 1587, 0, 1419, 0, 1242, 0, 1062, 0, 882, 0, 717, 0, 566, 0, 435, 0, 324, 0, 233, 0, 162, 0, 108, 0, 70, 0, 42, 0, 24, 0, 13, 0, 6, 0, 3, 0, 1.$

One can notice that $\{c_j^{(3)}(n)\}_{j\geq 0}$ has $3n(n+1)/2+1$ relevant terms, i.e., $c_j^{(3)}(n) = 0$ for $j > 3n(n+1)/2+1$. The rows of this triangle are not currently indexed in the encyclopedia of integer sequences OEIS.

D. *The case* $m = 4$

For $m = 4$ one recovers the sequence

$$c_0^{(4)}(n) : 6, 44, 426, 4658, 55260, 689508, 8914872, \ldots$$

indexed in OEIS as A124996. While several terms of this sequence have been computed, its asymptotic behaviour is not currently known. A conjecture based on numerical calculations is formulated in the following section.

For $m = 4$ and fixed values of n, the coefficients $\{c_j^{(4)}(n)\}_{j\geq 0}$ produce the following finite sequences:

$c_j^{(4)}(1):$ $6, 0, 4, 0, 1$

$c_j^{(4)}(2):$ $44, 0, 40, 0, 31, 0, 20, 0, 10, 0, 4, 0, 1$

$c_j^{(4)}(3):$ $426, 0, 408, 0, 372, 0, 320, 0, 251, 0, 188, 0, 130, 0, 80, 0, 47, 0, 24, 0, 10, 0, 4, 0, 1$

$c_j^{(4)}(4):$ $4658, 0, 4584, 0, 4380, 0, 4064, 0, 3650, 0, 3176, 0, 2680, 0, 2184, 0, 1716, 0, 1304, 0, 952, 0, 664, 0, 445, 0, 284, 0, 170, 0, 96, 0, 51, 0, 24, 0, 10, 0, 4, 0, 1$

$c_j^{(4)}(5):$ $55260, 0, 54792, 0, 53433, 0, 51236, 0, 48302, 0, 44768, 0, 40773, 0, 36492, 0, 32078, 0, 27692, 0, 23459, 0, 19492, 0, 15882, 0, 12672, 0, 9902, 0, 7564, 0, 5642, 0, 4108, 0, 2912, 0, 2008, 0, 1342, 0, 868, 0, 541, 0, 324, 0, 186, 0, 100, 0, 51, 0, 24, 0, 10, 0, 4, 0, 1$

One can notice that $\{c_j^{(4)}(n)\}_{j\geq 0}$ has $2n(n+1)+1$ relevant terms, i.e., $c_j^{(4)}(n) = 0$ for $j > 2n(n+1)+1$. The rows of this triangle are not currently indexed in the encyclopedia of integer sequences OEIS.

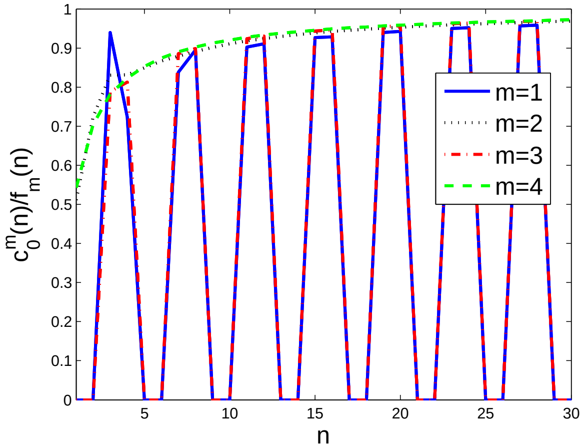

Fig. 1. First 30 terms of sequences $\frac{c_0^{(m)}(n)}{f_m(n)}$ evaluated for $m = 1, 2, 3, 4$, where $f_m(n) = \sqrt{\frac{6}{m\pi}}\frac{2^{mn}}{n\sqrt{n}}$.

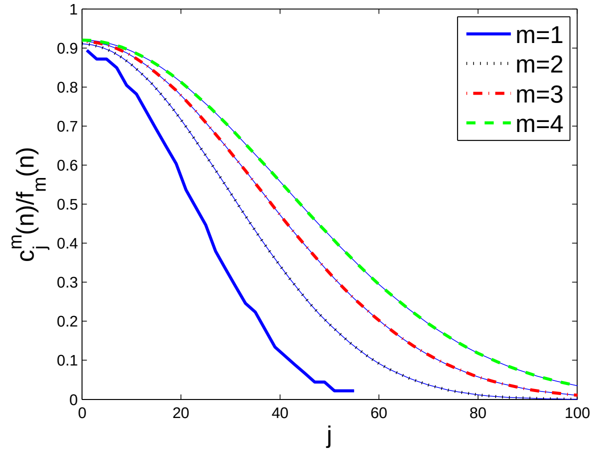

Fig. 2. Plot of sequences $\frac{c_j^{(m)}(n)}{f_m(n)}$ evaluated for $n = 10$, $m = 1, 2, 3, 4$ and $j \in \{0, \dots \frac{mn(n+1)}{2} + 1\}$. Function is given by $f_m(n) = \sqrt{\frac{6}{m\pi}}\frac{2^{mn}}{n\sqrt{n}}$.

V. Future work and conjectures

Numerical evidence suggests a number of conjectures.

A. Asymptotic behaviour of sequences $\{c_0^{(m)}(n)\}_{n\geq 0}$

In 2002, Andrica and Tomescu conjectured in the paper [3], the following asymptotic formula for $S(n) = c_0^{(1)}(n)$:

$$\lim_{\substack{n\to\infty \\ n\equiv 0 \text{ or } 3(mod 4)}} \frac{S(n)}{\frac{2^n}{n\sqrt{n}}} = \sqrt{\frac{6}{\pi}}.$$

Called the Andrica-Tomescu conjecture, this was proved in 2013 by using analytic methods by B.D. Sullivan [11].

The following asymptotic formula for $c_0^{(2)}(n)$ was then conjectured by Kotesovec in 2014 (see A047653 in [9]):

$$\lim_{n\to\infty} \frac{c_0^{(2)}(n)}{\frac{4^n}{n\sqrt{n}}} = \sqrt{\frac{3}{\pi}}.$$

We aim to establish a more general result for $c_0^{(m)}(n)$. The numerical results obtained for $m = 1, 2, 3, 4$ shown in Fig. 1, suggest the following asymptotic behaviour of $c_0^{(m)}(n)$.

Conjecture 1. Let $m \geq 1$ be an integer. The following asymptotic formula holds

$$\lim_{n\to\infty} \frac{c_0^{(m)}(n)}{\frac{2^{mn}}{n\sqrt{n}}} = \sqrt{\frac{6}{m\pi}}. \tag{14}$$

B. On the properties of $c_j^m(n)$

One may note that $\{c_j^{(m)}(n)\}_{j\geq 0}$ has $\frac{mn(n+1)}{2} + 1$ relevant terms, i.e., $c_j^{(m)}(n) = 0$ for $j > \frac{mn(n+1)}{2} + 1$, and we know that $\{c_j^{(1)}(n)\}_{j\in\mathbb{Z}}$ has a modulo 2 unimodality. Numerical evidence obtained so far suggests two other conjectures.

Conjecture 2. For any integers $m, n \geq 1$, the non-zero subsequence of $\{c_j^{(m)}(n)\}_{j\in\mathbb{Z}}$ is unimodal.

As $c_j^{(m)}(n) = c_{-j}^{(m)}(n)$ and every second term is zero, it is sufficient to prove that the subsequences $\{c_{2j}^{(m)}(n)\}_{j\geq 0}$ and $\{c_{2j+1}^{(m)}(n)\}_{j\geq 0}$ are decreasing, by using formula (6).

The following behaviour is suggested by Fig 2.

Conjecture 3. For any integers $m, n \geq 1$, the coefficients $c_j^{(m)}(n)$ belong to a normally distribution shaped curve.

If true, this conjecture would explain the unimodality of $c_j^{(m)}(n)$ for fixed m, n, as well as the asymptotic limits for $c_0^m(n)$ as n increases to infinity, for fixed m.

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